THEORY OF "HOT" CRACKS

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During the solidification of a welding seam, there sometimes develop in it so-called "hot" cracks, leading to a defect in the article. An analogous phenomenon of the formation of "pits" and voids in ingots is observed during the metallurgical process. There is considered below a theoretical model, within whose framework the problem of the formation and development of a hot crack can be solved. The solution of this problem permits comparing different thermal conditions and selecting the most favorable. A statement of the problem is given and the fundamental assumptions are formulated. A study is made of the kinetics of the growth of a hot crack. The question of the asymptotic dimension of hot cracks with $t \to \infty$ is discussed, and simple sufficient conditions are given, with whose satisfaction a hot crack is not formed. A study is made of the development of a crack in the mathematically similar problem of brittle failure from local heating.

1. Statement of Problem

At the initial moment of time, in contact with a solid metal having some constant temperature $T=0$, let there be a melt which solidifies instantaneously, so that, at the initial moment, its temperature is constant and equal to $T=T_0$. As a result of the solidification of the hot metal, elongational stresses develop in the region occupied by it, since, at the contact boundary, the metals are assumed to be rigidly welded. With the passage of time, the elongational stresses rise, bringing about a growth of the initially most dangerous crack or of some equivalent defect. With $t \to \infty$, the residual stresses and the dimension of the hot crack will be maximal.

We shall assume that the metals are thermoelastic bodies, so that all the plastic effects will be concentrated only in small regions near the contour of the crack. In this case, the problem posed with respect to the development of a hot crack can be solved within the framework of the mechanics of brittle failure [1].

We introduce also the following assumptions: a) all the thermoelastic constants are independent of the temperature and are identical for both cold and hot metal; b) the metals are homogeneous and isotropic bodies; c) these metals are in a plane state of stress (a thin plate). These assumptions are not of a fundamental character; however, they permit finding a simple effective solution to many practically important problems, and bringing out some of the fundamental qualitative effects. As is well known, the solutions obtained can be used also for the case of plane deformation, if the elastic coefficients are replaced.

Let us formulate a simplified problem. At the moment $t=0$, let an arbitrary region S in an infinite homogeneous and isotropic elastic plate be instantaneously heated to a constant temperature $T=T_0$. The remaining part of the body has the temperature $T=0$ with $t=0$. At the boundary of the region S, there is no discontinuity of the displacement; this corresponds physically to a replacement of the region S by a heated disk of exactly the same dimensions. It is required to determine the development of the initial crack with time. The displacements, the stresses, and the principal vector of the forces (as well as the rotation) at an infinitely distant point are assumed equal to zero.

2. Kinetics of the Growth of a Hot Crack

In the above statement of the problem, let the region S be a rectangle with the sides $2x_0$ and $2y_0$. We take the origin of the Cartesian coordinates x and y at the center of the rectangle, and we direct the x

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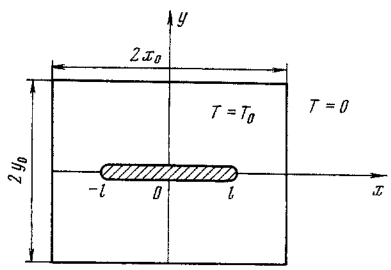

Fig. 1

ζ axis parallel to that side whose length is equal to 2 x_0 (Fig. 1). Let an initial crack of length 2 l be disposed along the x axis, with its center at the origin of coordinates. The sides of the crack are free of external loads. For the present problem, with an accuracy of approximately 10%, the coefficient of the intensity of the stresses at the end of the crack holds also for the case where the boundary of the body is free of external loads along the y axis.

The order of the solution of the problem will be the following. The temperature field is determined first; then, the stress σ_y with $y=0$ $|x| < l$ is found from the equations of thermoelasticity for a body without a crack; this stress, with the opposite sign, is substituted into the well-known general expression for the coefficient of the intensity of the stresses in the case of an isolated crack and of an isothermal process. The dependence of the constant K on the temperature outside of the interval of cold brittleness can be interpolated using the following linear function:

$$K_c = K_{c0} + AT\,(l,\,0,\,t) \tag{2.1}$$

where K_{c0} is the constant K_c with $T=0$; A is some empirical constant. Equating $K_I=K_c$, in accordance with the Griffiths–Irvin condition we obtain in implicit form the sought dependence of the length of the crack on the time.

The solution of the boundary-value problem

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = \frac{1}{a}\frac{\partial T}{\partial t} \quad (-\infty < x,\, y < +\infty)$$
$$T = \begin{cases} T_0 = \text{const} & (x,\, y \in S) \\ 0 & (x,\, y \notin S) \end{cases} \quad \text{with } t = 0 \tag{2.2}$$

will be the following [2]:

$$T\,(x,\,y,\,t) = \frac{T_0}{4}\left[\operatorname{Erf}\left(\frac{x+x_0}{2\sqrt{at}}\right) + \operatorname{Erf}\left(\frac{x_0-x}{2\sqrt{at}}\right)\right]\left[\operatorname{Erf}\left(\frac{y+y_0}{2\sqrt{at}}\right) + \operatorname{Erf}\left(\frac{y_0-y}{2\sqrt{at}}\right)\right]$$
$$\operatorname{Erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z \exp(-u^2)\,du \tag{2.3}$$

where a is the coefficient of thermal diffusivity.

The coefficients of the stress tensor σ_x, σ_y, τ_{xy} are expressed in terms of the thermoelastic potential of the displacements Ψ in the form [2]

$$\sigma_x = -2G\frac{\partial^2\Psi}{\partial y^2},\quad \sigma_y = -2G\frac{\partial^2\Psi}{\partial x^2},\quad \tau_{xy} = 2G\frac{\partial^2\Psi}{\partial x\,\partial y} \tag{2.4}$$

$$\Delta\Psi = (1+\nu)\alpha T \tag{2.5}$$

where G is the shear modulus; ν is the Poisson coefficient; α is the coefficient of linear expansion.

Differentiating (2.5) with respect to t, and taking account of (2.2), we obtain

$$\Delta[\partial\Psi / \partial t - (1+\nu)\alpha a T] = 0 \tag{2.6}$$

As can be seen, the function $\partial\Psi/\partial t - (1+\nu)\alpha a T$ is harmonic over the whole plane and, consequently, can be either a constant quantity or some function of the time g (t). With destroying the generality, the function g (t) can be assumed equal to zero, since, instead of the potential Ψ, the following potential can be introduced

$$\Psi' = \Psi - \int_0^t g(\tau)\,d\tau$$

Thus, an equation is obtained for the potential

$$\partial\Psi / \partial t = (1+\nu)\alpha a T$$

Integrating it leads to the formula

$$\Psi = (1+\nu)\alpha a \int_0^t T dt + \Psi_0(x, y) \tag{2.7}$$

where Ψ_0 (x, y) is the potential of the displacements corresponding to the initial temperature, i.e.,

$$\Delta\Psi_0 = (1+\nu)\alpha T_0 \text{ inside of } S, \ \Delta\Psi_0 = 0 \text{ outside of } S.$$

From this we obtain

$$\Psi_0(x, y) = -\frac{(1+\nu)\alpha T_0}{2\pi}\iint_S \ln\left(\frac{1}{R}\right) d\xi d\eta, \quad R = \sqrt{(x-\xi)^2 + (y-\eta)^2} \tag{2.8}$$

Using (2.3), (2.7), (2.8), and (2.4) we find

$$\sigma_y = -\frac{GT_0(1+\nu)\alpha}{4\sqrt{\pi}}\left\{4\sqrt{\pi}\varkappa(x, y) + \frac{4}{\sqrt{\pi}}\left[\operatorname{arc\,tg}\left(\frac{y+y_0}{x+x_0}\right) + \operatorname{arc\,tg}\left(\frac{y_0-y}{x_0-x}\right)\right.\right.$$
$$\left. + \operatorname{arc\,tg}\left(\frac{y_0-y}{x_0+x}\right) + \operatorname{arc\,tg}\left(\frac{y_0+y}{x_0-x}\right)\right] - \int_0^t \frac{1}{\tau\sqrt{a\tau}}\left[(x+x_0)\exp\left(-\frac{(x+x_0)^2}{4a\tau}\right) + (x_0-x)\exp\left(-\frac{(x_0-x)^2}{4a\tau}\right)\right]\left[\operatorname{Erf}\left(\frac{y+y_0}{2\sqrt{a\tau}}\right) + \tag{2.9}$$
$$\left.\left.\operatorname{Erf}\left(\frac{y_0-y}{2\sqrt{a\tau}}\right)\right] d\tau\right\} \quad \varkappa(x, y) = \begin{cases} 1 & (x, y \in S) \\ 0 & (x, y \notin S) \end{cases}$$

Since the coefficient of the intensity of the stresses for an isolated crack is

$$K_{\mathrm{I}} = -\frac{1}{\sqrt{\pi l}}\int_{-l}^{l} \sigma_y(x, 0, t)\sqrt{\frac{l+x}{l-x}}\, dx$$

then, using (2.9), we find

$$K_{\mathrm{I}} = \frac{GT_0(1+\nu)a}{\sqrt{\pi l}}\left\{\pi l + \frac{2}{\pi}\int_{-l}^{l}\left[\operatorname{arc\,tg}\left(\frac{y_0}{\xi+x_0}\right) + \operatorname{arc\,tg}\left(\frac{y_0}{x_0-\xi}\right)\right]\sqrt{\frac{l+\xi}{l-\xi}}\, d\xi - \right.$$
$$\left. - \frac{1}{2\sqrt{\pi}}\int_{-l}^{l}\int_0^t \frac{1}{x\sqrt{ax}}\left[(\xi+x_0)\exp\left(-\frac{(\xi+x_0)^2}{4ax}\right) + (x_0-\xi)\exp\left(-\frac{(x_0-\xi)^2}{4ax}\right)\right] \times \operatorname{Erf}\left(\frac{y_0}{2\sqrt{ax}}\right)\sqrt{\frac{l+\xi}{l-\xi}}\, dx\, d\xi\right\} \tag{2.10}$$

With t → ∞, we have

$$K_{\mathrm{I}} = GT_0(1+\nu)\alpha\sqrt{\pi l} \tag{2.11}$$

In accordance with (2.11) the length of the hot crack formed is

$$l_\infty = \frac{K_c^2}{\pi[GT_0(1+\nu)\alpha]^2} \text{ with } t \to \infty \tag{2.12}$$

It is assumed that the crack does not go beyond the region S.

For any given moment of time, on the basis of the criterion of local failure, $K_{\mathrm{I}} = K_c$, and (2.1), (2.10), we find

$$K_{c0}\left\{1 + \frac{AT_0}{2K_{c0}}\operatorname{Erf}\left(\frac{y_0}{2\sqrt{at}}\right)\left[\operatorname{Erf}\left(\frac{l+x_0}{2\sqrt{at}}\right) + \operatorname{Erf}\left(\frac{x_0-l}{2\sqrt{at}}\right)\right]\right\} = K_{\mathrm{I}}(l, t) \tag{2.13}$$

Here the first part is determined by formula (2.10). Figure 2, in the dimensionless variables $l_* = l/l_\infty$ and $t_* = 4at/x_0^2$, gives curves 1, 2, 3 for the kinetics of the growth of a hot crack, plotted using Eqs. (2.10) and (2.13), for the following values of the parameters: $x_0 = y_0 = l_\infty$, $A = K_{c0}/T_0$, $K_{c0}/2T_0$, 0, respectively. We note that the calculations were made in a BÉSM-4 digital computer in a few minutes.

3. Asymptotic Dimension of Hot Cracks

We first prove a theorem from the theory of thermoelasticity for an arbitrary singly connected region S.

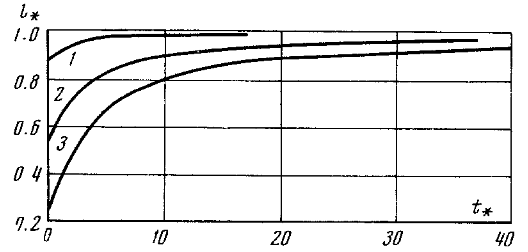

Fig. 2

THEOREM. With $t=0$ let

$$T(x, y, 0) = T_0 \text{ inside of } S,\ T(x, y, 0) = 0 \text{ outside of S.} \quad (3.1)$$

Then, with $t \to \infty$ $\sigma_y = \sigma_x = -G(1+\nu)\alpha T_0$, $\tau_{x,y} = 0$ inside of S.

Proof. The solution of the equation of thermal conductivity (2.2) satisfying conditions (3.1) has the form

$$T(x, y, t) = \frac{T_0}{4\pi at}\iint\limits_S \exp(-R^2/4at)\, d\xi\, d\eta,\ R^2 = (x-\xi)^2 + (y-\eta)^2 \tag{3.2}$$

Analogously to what has gone before, using (2.5)-(2.8) we find the potential

$$\Psi(x, y, t) = \frac{(1+\nu)\alpha T_0}{4\pi}\left\{\int\limits_0^t \frac{1}{\tau}\left[\iint\limits_S \exp(-R^2/4a\tau)\, d\xi\, d\eta\right] d\tau - 2\iint\limits_S \ln\left(\frac{1}{R}\right) d\xi\, d\eta\right\} \tag{3.3}$$

Since the contour of the region S does not depend on the time t, we obtain

$$\Psi(x, y, t) = \frac{(1+\nu)\alpha T_0}{4\pi}\iint\limits_S\left[\int\limits_0^t \frac{1}{\tau}\exp(-R^2/4a\tau)\, d\tau - 2\ln\left(\frac{1}{R}\right)\right] d\xi\, d\eta \tag{3.4}$$

Using (2.4) we find the stresses inside of S

$$\sigma_y = -G(1+\nu)\alpha T_0 - \frac{G(1+\nu)\alpha T_0}{2\pi}\iint\limits_S\left\{2\,\frac{(y-\eta)^2-(x-\xi)^2}{R^4} + \int\limits_0^t \frac{1}{4a^2\tau^3}[(x-\xi)^2 - 2a\tau]\exp(-R^2/4a\tau)\, d\tau\right\} d\xi\, d\eta$$

$$\sigma_x = -G(1+\nu)\alpha T_0 - \frac{G(1+\nu)\alpha T_0}{2\pi}\iint\limits_S\left\{2\,\frac{(x-\xi)^2-(y-\eta)^2}{R^4} + \int\limits_0^t \frac{1}{4a^2\tau^4}(y-\eta)^2 - 2a\tau]\exp(-R^2/4a\tau)\, d\tau\right\} d\xi\, d\eta$$

$$\tau_{xy} = -\frac{2G(1+\nu)\alpha T_0}{\pi}\iint\limits_S \frac{(x-\xi)(y-\eta)}{R^4}\left[1 - \frac{R^4}{16a^2}\int\limits_0^t \frac{1}{\tau^3}\exp(-R^2/4a\tau)\, d\tau\right] d\xi\, d\eta \tag{3.5}$$

Calculating the internal integrals in formulas (3.5) we have

$$\sigma_y = -G(1+\nu)\alpha T_0\left\{1 + \frac{1}{\pi}\iint\limits_S \frac{1}{R^4}[(y-\eta)^2 - (x-\xi)^2 + 2(x-\xi)^2\Gamma(2, R^2/4at) - R^2\exp(-R^2/4at)]\, d\xi\, d\eta\right\}$$

$$\sigma_x = -G(1+\nu)\alpha T_0\left\{1 + \frac{1}{\pi}\iint\limits_S \frac{1}{R^4}[(x-\xi)^2 - (y-\eta)^2 + 2(y-\eta)^2\Gamma(2, R^2/4at) - R^2\exp(-R^2/4at)]\, d\xi\, d\eta\right\}$$

$$\tau_{xy} = -\frac{G(1+\nu)\alpha T_0}{2\pi}\iint\limits_S \frac{4(x-\xi)(y-\eta)}{R^4}[1 - \Gamma(2, R^2/4at)]\, d\xi\, d\eta$$

$$\Gamma(\alpha, x) \equiv \int\limits_x^\infty e^{-t}t^{\alpha-1}\, dt \tag{3.6}$$

Since $\Gamma(2, R^2/4at) \to 1$ with $t \to \infty$, then, passing to the limit with $t \to \infty$ in (3.6), we obtain

$$\sigma_y = \sigma_x = -G(1+\nu)\alpha T_0,\ \tau_{xy} = 0 \tag{3.7}$$

Thus, in the region S, with $t \to \infty$ the residual stresses create a state of isotropic uniform elongation. We note that bilateral elongation promotes brittle failure to a greater degree than unilateral elongation [1].

The theorem proved permits finding the asymptotic dimension l_∞ of a hot crack with $t \to \infty$ for the arbitrary region S; this dimension, as before, is obviously given by formula (2.12), under the sole condition that the crack does not go beyond the region S. It follows from this that, if the length of the initial crack $2l_0$ is smaller than $2l_\infty$, i.e., if the following condition is satisfied

$$l_0 < K_c^2 / \pi[GT_0(1+\nu)\alpha]^2 \tag{3.8}$$

then, the initial crack does not develop. Formula (3.8) furnishes a simple sufficient criterion; when it is satisfied, a hot crack is not formed. We note that the characteristics of the material l_0 and K_0 in (3.8) correspond to solidified metal with $T=0$.

4. Brittle Failure From Local Heating

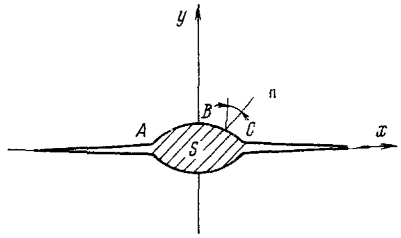

Fig. 3

With t = 0, let a discontinuity of the normal displacement arise at the boundary of the region S, corresponding to the instantaneous heating of the disk S up to the temperature T_0, if, before heating, the disk was inserted without clearance (the temperature of the remaining part of the body is equal to zero with t = 0). As a result, elongational stresses arise in the body outside of the region S, which can lead to the formation of a crack. The stresses in the disk will be compressive. With $t \to \infty$, in this case the stresses in the whole body will obviously tend to zero. With t = 0, the stresses will be greatest; here a state of isotropic uniform compression is set up in the disk

$$\sigma_x = \sigma_y = -G(1+\nu)\alpha T_0, \ \tau_{xy} = 0 \text{ with } t = 0 \tag{4.1}$$

It is evident that in this case a crack is formed instantaneously, and then does not develop further.

The above-described mechanism of brittle failure is typical for the local heating of an unbroken material (for example, the formation of a crack in a thick-walled glass when hot water is poured into it suddenly).

In a majority of practically important cases, for brittle materials the length of a crack forming as a result of local heating is far greater than the characteristic linear dimension of the region S. In this latter region, a simple asymptotic method, based on the "microscope principle" can be used to solve the problem [1].

We postulate that at the points A and C cracks go out toward the contour of the region S (Fig. 3). For simplicity we assume that the region S is symmetrical with respect to the x and y axes. Stresses (from the side of the disk) determined by formula (4.1) will be applied to the boundary of the region S.

Let us determine the dimensions of asymptotically large cracks. On the basis of the microscope principle [1], used in this case, when the linear dimensions of the region S are small in comparison with the length of the crack (or with its radius, in the axisymmetric case), we arrive at the following singular problems.

Plane Problem. To the opposing sides of a rectilinear through crack of length $2l$, existing in an infinite plate, let there be applied the directed concentrated forces P; the force P acts in the middle of the crack, perpendicularly to its surface. At infinity there is no stress. Using (4.1), we find the principal vector of the loads applied to the arc ABC from the side of the heated region S (see Fig. 3)

$$P = G(1+\nu)\alpha T_0 \int_{ABC} \cos(n, y)\, ds = G(1+\nu)\alpha T_0 L \tag{4.2}$$

where L is the length of the projection of the arc ABC on the x axis.

In the case under consideration, the coefficient of the intensity of the stresses [1] is

$$K_I = P / \pi \sqrt{2l} \tag{4.3}$$

In accordance with the criterion of local failure, the length of a brittle crack is determined by the condition $K_I = K_c$, i.e.,

$$K_c = P / \pi \sqrt{2l} \tag{4.4}$$

From this we find the length of asymptotically large cracks

$$l = \frac{1}{2}\left[\frac{G(1+\nu)\alpha T_0 L}{\pi K_c}\right]^2 \tag{4.5}$$

Axisymmetric Problem. Let some region V, having three planes of symmetry, be heated instantaneously to a temperature T_0 at the initial moment of time. As a result of instantaneous brittle failure, a disk-shaped crack is formed and, in the most frequently encountered case, when its radius is great in comparison with the dimension of the region V, on the basis of the microscope principle [1] we arrive at the following problem.

To the opposing surfaces of a round disk-shaped crack of radius R, located in an infinite body, let there be applied the equal and oppositely directed forces P; the force P acts along the axis of the round disc-shaped crack. In this case, we obviously have

$$P = G(1+\nu)\alpha T_0 \iint (n_z dS) = G(1+\nu)\alpha T_0 S_0 \tag{4.6}$$

where S_0 is the area of the projection of the boundary of the region V on the plane of the crack.

The coefficient of the intensity of the stresses is [1]

$$K_I = P / (\pi R)^{3/2} \tag{4.7}$$

Using the criterion of local failure $K_I = K_{Ic}$, we find the radius of the disk-shaped crack

$$R = \pi^{-1} [G(1+\nu)\alpha T_0 S_0 / K_{Ic}]^{2/3} \tag{4.8}$$

For example, in the case of an ellipsoidal region V, we have

$$S_0 = 4ab, \;\; R = \pi^{-1} [4G(1+\nu)\alpha ab T_0 / K_{Ic}]^{2/3} \tag{4.9}$$

Here a and b are the principal semi-axes of the elliple in a cross section of the ellipsoid of a disk-shaped crack.

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